JOURNAL OF INTERNATIONAL ISLAMIC LAW, HUMAN RIGHT AND PUBLIC POLICY

Journal of International Islamic Law, Human Rights and Public Policy (JISHUP Journal) | ISSN(e): 3031-2280
Volumes 3 No. 2 (2025)

Laplace transform and its applications to Fractional differential equations

Muneer Ahmad Sofi[1] , Sobiya Jan[2]
[1]Research scholar, Department of Mathematics Sant Baba Bhag Singh University, Jalandhar-144030, Punjab, India
[2]M.sc Mathematics, Department of Mathematics Baba Ghulam Shah Badshah University Rajouri-185234, India

Received	: 25 February 2025	Published	: 29 April 2025
Revised	: 05 March 2025	DOI	: https://doi.org/10.59733/jishup.v3i2.138
Accepted	: 27 March 2025	Publish Link	: https://jishup.org/index.php/ojs

Abstract

This paper investigates the application of the Laplace transform method in solving fractional differential equations. It establishes sufficient conditions under which the Laplace transform provides a rational approach to these problems. Key definitions and properties of fractional calculus, including the Riemann-Liouville and Caputo fractional derivatives, are discussed. Several lemmas are proved to facilitate the computation of inverse Laplace transforms involving fractional operators. The effectiveness of the method is demonstrated through examples of solving linear fractional differential equations with exact solutions. The study concludes that while the Laplace transform is well-suited for fractional differential equations with constant coefficients, its applicability is limited by the nature of the forcing terms.

Keywords: ***Laplace transform, Riemann- Liouville fractional derivative Caputo fractional derivative, Fractional differential equations.***

Introduction

This paper deals with the rationality of Laplace transform for solving the fractional differential equation. The main benefit of fractional derivatives over integer-order derivatives is that they can describe the memory and heredity properties of various materials. Differential equations of arbitrary (non integer) order are known as fractional differential equations. Because fractional differential equations have so many uses in science and engineering, they have garnered a lot of attention lately. See the books of Podlubny [1], and the paper of Ahmad and Sivasundaram [4].

Preliminaries

We provide some basic definitions,lemma and properties of the theory of fractional calculus that are further us ed in this article.

Definition 1.1

The Laplace transform $F(s)$ is defined as a function $f(x)$ for $0 < x < 1$.

$$F(s) = L[f(t)] = \int_0^\infty f(x)e^{-sx}dx \tag{1}$$

For that s at least for which the integral converges.

Definition 1.2: A real function $f(x), x > 0$ is said to be in space $C_\mu, \mu \in \mathbb{R}$ if there is a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C(0,\infty)$, and if and only if $f^n \in C_\mu, n \in N$ it is said to be in space C_μ^n.

Definition 1.3: The fractional integral operator of the Riemann Liouville order $\alpha > 0$, of the function $f \in C_\mu, \mu \geq -1$, is defined as

$$j^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds, \quad \alpha > 0 \tag{2}$$

Some of the operator's j^α, properties required in this case are as follows:

If $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma \geq -1$.

(1) $j^\alpha j^\beta f(t) = j^{\alpha+\beta} f(t)$

(2) $j^{\alpha}t^{\gamma}=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}t^{\alpha+\gamma}$ (3)

Definition 1.4: The Caputo fractional derivative of $f\ (t)$ is defined as

$$D^{\alpha}f(t)=j^{m-\alpha}D^{m}f(t) \tag{4}$$

For $m-1<\alpha\leq m, m\in\mathbb{N}, t>0\ and\ f\in C_{-1}^{m}$.

Caputo fractional derivative calculates a common derivative first, followed by a fractional integral to achieve the desir ed fractional derivative order.

The Riemann-Liouville fractional integral operator is a linear operation, similar to the integration of the integer-order:

$$j^{\alpha}\left(\sum_{i=1}^{n}c_i\ f_i(t)\quad\right)=\sum_{i=1}^{n}c_i j^{\alpha}\ f_i(t) \tag{5}$$

Where $\{c_i\}_{i=1}^{n}$ are constants.

Lemma 1 The Riemann-Liouville fractional integral operator Laplace transformation of order α > 0 can be obtained as:

$$L[j^{\alpha}f(x)]=\frac{F(s)}{s^{\alpha}}$$

Proof:

The Riemann-Liouville fractional integral operator Laplace transformation of order α > 0 can be obtained as:

$$L[j^{\alpha}f(x)]=L\left[\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}f(t)dt\right]$$
$$=\frac{1}{\Gamma(\alpha)}F(s)G(s)$$

Where

$$G(s)=L[x^{\alpha-1}]=\frac{\Gamma(\alpha)}{s^{\alpha}}$$

Lemma 2 It is possible to obtain the Laplace transformation of Caputo fractional derivative for $m-1<\alpha\leq m, m\in\mathbb{N}$, in the form of:

$$L[D^{\alpha}f(x)]=\frac{s^{m}F(s)-s^{n-1}f(0)-s^{m-2}f'^{(0)}-\cdots\ldots\ldots\ldots\ldots\ldots.-f^{m-1}(0)}{s^{m-\alpha}}$$

Proof:

The Caputo fractional derivative Laplace transformation of order $\alpha>0$ is:

$$L[D^{\alpha}f(x)]=L[j^{m-\alpha}f^{m}(x)]=L\left[\frac{f^{m}(x)}{s^{m-\alpha}}\right]$$

Use equation (.3). Now we can convert fractional differential equations into algebraic equations and then we can obtain the unknown Laplace function F(s) by solving these algebraic equations.

Inverse Laplace transform

The $f(x)$ function in (1) is called the $F(s)$ inverse Laplace transform and is denoted in the paper as $f(x)=L^{-1}[F(s)]$. in practice, for example, when you use the Laplace transform to solve a differential equation, you have to invert the Laplace transform at some point by finding the function $f(x)$ that corresponds to certain defined $F(s)$.

The F(s) transformation of the Inverse Laplace is defined as:

$$f(x)=L^{-1}[F(s)]=\frac{1}{2\pi i}\log_{T\to\infty}\int_{\sigma-iT}^{\sigma+T}e^{sx}F(s)ds$$

Where σ is sufficiently large to describe $F(s)$ for the actual part of $s\geq\sigma$, surprisingly, this formula is not really useful. Consequently, some useful function $f(x)$ from their Laplace transform is obtained in this section. In the first, we define the most important special functions used in fractional calculus, the functions of Mittag-Leffler and the generalized functions of Mittag-Leffler.

For $\alpha,\beta>0\ and\ z\in\mathbb{C}$

$$E_{\alpha}(z)=\sum_{n=0}^{\infty}\frac{z^{n}}{\Gamma(n\alpha+1)}$$

Published by **Radja Publika**

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)}$$

Now we're proving some Lemmas that are useful in finding the $f(x)$ function from its transform Laplace.

Lemma 3 we have the following inverse equation for $\alpha,\ \beta\ >\ 0, a \in \mathbb{R}\ and\ s^{\ \alpha} > |a|$.

$$L^{-1}\left[\frac{s^{\alpha-\beta}}{s^{\ \alpha}+a}\right] = x^{\beta-1}E_{\alpha,\beta}(-ax^{\alpha})$$

Proof: $s^{\alpha-\beta}\Big/_{s^{\ \alpha}+a}$ in series expansion it can be written as

$$\frac{s^{\alpha-\beta}}{s^{\ \alpha}+a} = \frac{1}{s^{\beta}}\frac{1}{1+\frac{a}{s^{\ \alpha}}} = \frac{1}{s^{\beta}}\sum_{n=0}^{\infty}(\frac{-a}{s^{\ \alpha}})^n$$

$$= \sum_{n=0}^{\infty}\frac{(-a)^n}{s^{n\alpha+\beta}}$$

The inverse transformation of the Laplace function above is

$$\sum_{n=0}^{\infty}\frac{(-a)^n x^{n\alpha+\beta-1}}{\Gamma(n\alpha+\beta)} = x^{\beta-1}\sum_{n=0}^{\infty}\frac{(-ax^{\alpha})^n}{\Gamma(n\alpha+\beta)}$$

$$= x^{\beta-1}E_{\alpha,\beta}(-ax^{\alpha})^n$$

Lemma 4 for $\alpha \geq\ \beta\ >\ 0, a \in \mathbb{R}\ and\ s^{\ \alpha-\beta} > |a|$.then

$$L^{-1}\left[\frac{1}{(s^{\alpha}+as^{\beta})^{n+1}}\right] = x^{\alpha(n+1)-1}\sum_{n=0}^{\infty}\frac{(-a)^k\binom{n+k}{k}}{\Gamma\big(k(\alpha-\beta)+n(\alpha+1)\big)}x^{k(\alpha-\beta)}$$

Proof:

Using the expansion series of $(1+x)^{-n-1}$

$$\frac{1}{(1+x)^{n+1}} = \sum_{n=0}^{\infty}\binom{n+k}{k}(-x)^k$$

We get

$$\frac{1}{(s^{\alpha}+as^{\beta})^{n+1}} = \frac{1}{(s^{\alpha})^{n+1}}\frac{1}{\left(1+\frac{\alpha}{s^{\alpha-\beta}}\right)}$$

$$= \frac{1}{(s^{\alpha})^{n+1}}\sum_{n=0}^{\infty}\binom{n+k}{k}\left(\frac{-a}{s^{\alpha-\beta}}\right)^k$$

The Lemma will prove to be the inverse Laplace transforms of the above function.

Lemma 5 for $\alpha \geq\ \beta\ , \alpha > \gamma, a \in \mathbb{R}\ ,\ s^{\ \alpha-\beta} > |a| and \left|\ s^{\ \alpha} + a\ s^{\ \beta}\right| > |b|$ then

$$L^{-1}\left[\frac{s^{\gamma}}{s^{\alpha}+as^{\beta}+b}\right] = x^{\alpha-\gamma-1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-b)^n(-a)^k\binom{n+k}{k}}{\Gamma(k(\alpha-\beta)+(n+1)\alpha-\gamma)}x^{k(\alpha-\beta)+n\alpha}$$

Proof: $s^{\gamma}\Big/_{s^{\alpha}+as^{\beta}+b}$ in series expansion it can be written as

$$\frac{s^{\gamma}}{s^{\alpha}+as^{\beta}+b} = \frac{s^{\gamma}}{s^{\alpha}+as^{\beta}}\frac{1}{1+\frac{b}{s^{\alpha}+as^{\beta}}} = \frac{1}{s^{\beta}}\sum_{n=0}^{\infty}\frac{s^{\gamma}(-b)^n}{(s^{\alpha}+as^{\beta})^{n+1}}$$

Now it is possible to prove the Lemma by using Lemma 4.

Applications to fractional differential equations

This section applies the method described in the paper and provides some linear fractional differential equations with an exact solution.

Example 1: For the inhomogeneous Bagley-Torvik equation, we consider the following initial value question as to the first example.

$$D^2y(x) + D^{\frac{3}{2}}y(x) + y(x) = 1 + x \tag{6}$$
$$y(0) = y'(0) = 1$$

Taking Laplace transform on both sides of equation (6), we get

$$s^2F(s) - sy(0) - y'(0) + \frac{s^2F(s) - sy(0) - y'(0)}{s^{\frac{1}{2}}} + F(s) = \frac{1}{s} + \frac{1}{s^2}$$
$$s^2F(s) - s - 1 + \frac{s^2F(s) - s - 1}{s^{\frac{1}{2}}} + F(s) = \frac{1}{s} + \frac{1}{s^2}$$
$$F(s)\left(s^2 + s^{\frac{3}{2}} + 1\right) = \left(\frac{1}{s} + \frac{1}{s^2}\right)\left(s^2 + s^{\frac{3}{2}} + 1\right)$$
$$F(s) = \left(\frac{1}{s} + \frac{1}{s^2}\right)$$

The exact solution of this problem can be obtained using the inverse Laplace transform $y(x) = 1 + x$.

Example 2: Our second example is the linear equation inhomogeneous.

$$D^\alpha y(x) + y(x) = \frac{2x^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{x^{1-\alpha}}{\Gamma(2-\alpha)} + x^2 - x \tag{7}$$
$$y(0) = 0, \quad 0 < \alpha \leq 1$$

By using Laplace transform $F(s)$ can be obtained as

$$\frac{sF(s) - y(0)}{s^{1-\alpha}} = \frac{2}{s^{3-\alpha}} - \frac{1}{s^{2-\alpha}} - F(s) + \frac{2}{s^3} - \frac{1}{s^2}$$
$$F(s)(s^\alpha + 1) = 2\frac{s^{\alpha+1}}{s^3} - \frac{s^{\alpha+1}}{s^2}$$
$$F(s) = \frac{2}{s^3} - \frac{1}{s^2}$$

Using the inverse Laplace transform $y(x) = x^2 - x$ is then obtained.

Examples 3 consider the following initial value linear problem.

$$D^\alpha y(x) + y(x) = 0 \tag{8}$$

Only $\alpha > 1$ is the second initial condition.

$L\,[\,D^\alpha f(x)\,]$ Is obtained in two cases of α.

1. If $\alpha < 1$

$$L\,[\,D^\alpha f(x)] = \frac{s^2F(s) - s}{s^{2-\alpha}} = \frac{sF(s) - 1}{s^{1-\alpha}}$$

2. If $\alpha > 1$

$$L\,[\,D^\alpha f(x)] = \frac{sF(s) - 1}{s^{1-\alpha}}$$

Which are same. Now Laplace transform of F(s) is obtained as

$$\frac{sF(s) - 1}{s^{1-\alpha}} + F(s) = 0$$
$$F(s) = \frac{s^{\alpha-1}}{1 + s^\alpha}$$

The exact solution to this problem can be obtained using lemma 3 as:

$$y(x) = E_\alpha(-x)^\alpha$$

CONCLUSION

In this paper, The Laplace transform method is suitable for constant coefficient fractional differential equations, but it demands for forcing terms, so not every constant coefficient fractional differential equation can be solved by the Laplace transform method.

REFERENCES

1. I. Podlubny Fractional Differential Equations.
2. K.S. Miller, B. Boss An Introduction to the Fractional Calculus and Fractional Differential Equations John Wiley, New York (1993).
3. Z.M. Odibat Analytic study on linear systems of fractional differential equations Comput. Math. Appl., 59 (3) (2010), pp. 1171-1183.
4. B. Ahmad, S. Sivasundaram Existence results for nonlinear implusive hybrid boundary value problems involving fractional differential equations Nonlinear Anal. Hybrid Syst., 3 (3) (2009), pp. 251-258.
5. W. Lin Global existence and chaos control of fractional differential equations J. Math. Anal. Appl., 332 (2007), pp. 709-726.
6. K.B. Oldham, J. Spanier The Fractional Calculus Academic Press, New York-London (1974).
7. X.J. Wen, Zh.M. Wu, J.G. Lu Stability analysis of a class of nonlinear fractional-order systems IEEE Trans. Circuits Syst. II, Express. Briefs, 55 (11) (2008), pp. 1178-1182.
8. X. Huang and X. Lu, the Use of Fractional B-Splines Wavelets in Multiterms Fractional Ordinary Differential Equations, International Journal of Differential equations Vol.(2010).
9. Imam Gorial,I Semi-Analytic Method for Solving Two-Dimensional Fractional Dispersion Equation, Advances in Physics Theories and Applications,vol.22(2013).
10. L.Kexue and P.jigen, Laplace transform and fractional differential equations, Applied Mathematics Letters, 24,219-223(2011).
11. Nurettin Doğan, solution of the system of ordinary differential equations by combined Laplace transform-Adomian decomposition method, Mathematical and Computational Applications, Vol.(17)3, 203-211(2012).
12. S.Kazem, Exact solution of some linear fractional differential equations by Laplace transform, International journal of Nonlinear Science, 16(1), 3-11(2013).